Exercises in Algebraic Number Theory

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Sheet 4

To be handed in by: Friday, 13 November 2009, 2 pm.

1. (4 points) Let R be a ring. Show that R is a principal ideal domain if and only if R is a Dedekind ring with $\mathrm{Pic}(R) = 0$.

2. (4 points) Let R be a Noetherian integral domain of Krull dimension 1 and $I \trianglelefteq R$ be an invertible integral ideal. Let $\mathfrak{p}$ be a maximal ideal of R. Let $I_{(\mathfrak{p})} := I \cap R$ be the $\mathfrak{p}$-primary part of I.

   Show that I has a *primary decomposition*, i.e. $I = \bigcap_{\mathfrak{p} \supseteq I} I_{(\mathfrak{p})}$.

3. (4 points)

   (a) Let R be a discrete valuation ring with field of fractions K and maximal ideal $\mathfrak{p}$. Recall that we denote by $\mathrm{ord}_{\mathfrak{p}}(x)$ for $x \in K^\times$ the maximum integer n such that $x \in \mathfrak{p}^n$. Let $\mathrm{ord}_{\mathfrak{p}}(0) = \infty$. Show that the map
   $$v : K \to \mathbb{Z}, \quad x \mapsto \mathrm{ord}_{\mathfrak{p}}(x) \tag{0.1}$$
   satisfies
   $$v(x + y)) \geq \min(v(x), v(y)) \qquad \text{for all } x, y \in K. \tag{0.2}$$
   The map v is called a *discrete valuation*.

   (b) Let K be a field together with a discrete valuation v as in (0.1) satisfying (0.2). Show that
   $$R_v := \{x \in K \mid v(x) \geq 0\}$$
   is a discrete valuation ring. What is its maximal ideal?

   (c) Show that every discrete valuation ring is a valuation ring (see Sheet 2, Exercise 2).

4. (4 points) Let R be a Noetherian ring of Krull dimension zero (i.e. every prime ideal of R is a maximal ideal). Show that R is an Artinian ring, i.e. that every descending chain of ideals
   $$\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \mathfrak{a}_3 \supseteq \dots$$
   becomes stationary, i.e. there is $n \in \mathbb{N}$ such that $\mathfrak{a}_n = \mathfrak{a}_{n+i}$ for all $i \in \mathbb{N}$.

   Here is a possible strategy. You may use a different one. Let $\mathfrak{N}$ be the nilradical of R, i.e. the set of all nilpotent elements. It is the intersection of the prime ideals of R. Now consider the natural map $R \to \prod_{\mathfrak{P}} R/\mathfrak{P} =: E$. Show that E has only finitely many ideals and is hence an Artinian R-module. Now consider the quotients $\mathfrak{n}^m/\mathfrak{n}^{m+1}$ and conclude.